

## Chapter 39

1. According to Eq. 39-4,  $E_n \propto L^{-2}$ . As a consequence, the new energy level  $E'_n$  satisfies

$$\frac{E'_n}{E_n} = \left(\frac{L'}{L}\right)^{-2} = \left(\frac{L}{L'}\right)^2 = \frac{1}{2},$$

which gives  $L' = \sqrt{2}L$ . Thus, the ratio is  $L'/L = \sqrt{2} = 1.41$ .

2. (a) The ground-state energy is

$$\begin{aligned} E_1 &= \left(\frac{h^2}{8m_e L^2}\right) n^2 = \left(\frac{(6.63 \times 10^{-34} \text{ J} \cdot \text{s})^2}{8(9.11 \times 10^{-31} \text{ kg})(200 \times 10^{-12} \text{ m})^2}\right) (1)^2 = 1.51 \times 10^{-18} \text{ J} \\ &= 9.42 \text{ eV}. \end{aligned}$$

(b) With  $m_p = 1.67 \times 10^{-27} \text{ kg}$ , we obtain

$$\begin{aligned} E_1 &= \left(\frac{h^2}{8m_p L^2}\right) n^2 = \left(\frac{(6.63 \times 10^{-34} \text{ J} \cdot \text{s})^2}{8(1.67 \times 10^{-27} \text{ kg})(200 \times 10^{-12} \text{ m})^2}\right) (1)^2 = 8.225 \times 10^{-22} \text{ J} \\ &= 5.13 \times 10^{-3} \text{ eV}. \end{aligned}$$

3. Since  $E_n \propto L^{-2}$  in Eq. 39-4, we see that if  $L$  is doubled, then  $E_1$  becomes  $(2.6 \text{ eV})(2)^{-2} = 0.65 \text{ eV}$ .

4. We first note that since  $h = 6.626 \times 10^{-34} \text{ J} \cdot \text{s}$  and  $c = 2.998 \times 10^8 \text{ m/s}$ ,

$$hc = \frac{(6.626 \times 10^{-34} \text{ J} \cdot \text{s})(2.998 \times 10^8 \text{ m/s})}{(1.602 \times 10^{-19} \text{ J/eV})(10^{-9} \text{ m/nm})} = 1240 \text{ eV} \cdot \text{nm}.$$

Using the  $mc^2$  value for an electron from Table 37-3 ( $511 \times 10^3 \text{ eV}$ ), Eq. 39-4 can be rewritten as

$$E_n = \frac{n^2 h^2}{8mL^2} = \frac{n^2 (hc)^2}{8(mc^2)L^2}.$$

The energy to be absorbed is therefore

$$\Delta E = E_4 - E_1 = \frac{(4^2 - 1^2)h^2}{8m_e L^2} = \frac{15(hc)^2}{8(m_e c^2)L^2} = \frac{15(1240 \text{ eV} \cdot \text{nm})^2}{8(511 \times 10^3 \text{ eV})(0.250 \text{ nm})^2} = 90.3 \text{ eV}.$$

5. We can use the  $mc^2$  value for an electron from Table 37-3 ( $511 \times 10^3 \text{ eV}$ ) and  $hc = 1240 \text{ eV} \cdot \text{nm}$  by writing Eq. 39-4 as

$$E_n = \frac{n^2 h^2}{8mL^2} = \frac{n^2 (hc)^2}{8(mc^2)L^2}.$$

For  $n = 3$ , we set this expression equal to 4.7 eV and solve for  $L$ :

$$L = \frac{n(hc)}{\sqrt{8(mc^2)E_n}} = \frac{3(1240 \text{ eV} \cdot \text{nm})}{\sqrt{8(511 \times 10^3 \text{ eV})(4.7 \text{ eV})}} = 0.85 \text{ nm}.$$

6. With  $m = m_p = 1.67 \times 10^{-27} \text{ kg}$ , we obtain

$$E_1 = \left( \frac{h^2}{8mL^2} \right) n^2 = \left( \frac{(6.63 \times 10^{-34} \text{ J} \cdot \text{s})^2}{8(1.67 \times 10^{-27} \text{ kg})(100 \times 10^{12} \text{ m})^2} \right) (1)^2 = 3.29 \times 10^{-21} \text{ J} = 0.0206 \text{ eV}.$$

Alternatively, we can use the  $mc^2$  value for a proton from Table 37-3 ( $938 \times 10^6 \text{ eV}$ ) and  $hc = 1240 \text{ eV} \cdot \text{nm}$  by writing Eq. 39-4 as

$$E_n = \frac{n^2 h^2}{8mL^2} = \frac{n^2 (hc)^2}{8(m_p c^2)L^2}.$$

This alternative approach is perhaps easier to plug into, but it is recommended that both approaches be tried to find which is most convenient.

7. To estimate the energy, we use Eq. 39-4, with  $n = 1$ ,  $L$  equal to the atomic diameter, and  $m$  equal to the mass of an electron:

$$E = n^2 \frac{h^2}{8mL^2} = \frac{(1)^2 (6.63 \times 10^{-34} \text{ J} \cdot \text{s})^2}{8(9.11 \times 10^{-31} \text{ kg})(1.4 \times 10^{-14} \text{ m})^2} = 3.07 \times 10^{-10} \text{ J} = 1920 \text{ MeV} \approx 1.9 \text{ GeV}.$$

8. The frequency of the light that will excite the electron from the state with quantum number  $n_i$  to the state with quantum number  $n_f$  is

$$f = \frac{\Delta E}{h} = \frac{h}{8mL^2} (n_f^2 - n_i^2)$$

and the wavelength of the light is

$$\lambda = \frac{c}{f} = \frac{8mL^2c}{h(n_f^2 - n_i^2)}.$$

The width of the well is

$$L = \sqrt{\frac{\lambda hc(n_f^2 - n_i^2)}{8mc^2}}.$$

The longest wavelength shown in Figure 39-27 is  $\lambda = 80.78 \text{ nm}$ , which corresponds to a jump from  $n_i = 2$  to  $n_f = 3$ . Thus, the width of the well is

$$L = \sqrt{\frac{\lambda hc(n_f^2 - n_i^2)}{8mc^2}} = \sqrt{\frac{(80.78 \text{ nm})(1240 \text{ eV} \cdot \text{nm})(3^2 - 2^2)}{8(511 \times 10^3 \text{ eV})}} = 0.350 \text{ nm} = 350 \text{ pm}.$$

9. We can use the  $mc^2$  value for an electron from Table 37-3 ( $511 \times 10^3 \text{ eV}$ ) and  $hc = 1240 \text{ eV} \cdot \text{nm}$  by rewriting Eq. 39-4 as

$$E_n = \frac{n^2 h^2}{8mL^2} = \frac{n^2 (hc)^2}{8(mc^2)L^2}.$$

(a) The first excited state is characterized by  $n = 2$ , and the third by  $n' = 4$ . Thus,

$$\begin{aligned} \Delta E &= \frac{(hc)^2}{8(mc^2)L^2} (n'^2 - n^2) = \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8(511 \times 10^3 \text{ eV})(0.250 \text{ nm})^2} (4^2 - 2^2) = (6.02 \text{ eV})(16 - 4) \\ &= 72.2 \text{ eV}. \end{aligned}$$

Now that the electron is in the  $n' = 4$  level, it can “drop” to a lower level ( $n''$ ) in a variety of ways. Each of these drops is presumed to cause a photon to be emitted of wavelength

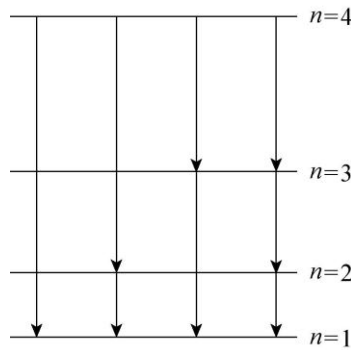
$$\lambda = \frac{hc}{E_{n'} - E_{n''}} = \frac{8(mc^2)L^2}{hc(n'^2 - n''^2)}.$$

For example, for the transition  $n' = 4$  to  $n'' = 3$ , the photon emitted would have wavelength

$$\lambda = \frac{8(511 \times 10^3 \text{ eV})(0.250 \text{ nm})^2}{(1240 \text{ eV} \cdot \text{nm})(4^2 - 3^2)} = 29.4 \text{ nm},$$

and once it is then in level  $n'' = 3$  it might fall to level  $n''' = 2$  emitting another photon. Calculating in this way all the possible photons emitted during the de-excitation of this system, we obtain the following results:

- (b) The shortest wavelength that can be emitted is  $\lambda_{4 \rightarrow 1} = 13.7 \text{ nm}$ .
- (c) The second shortest wavelength that can be emitted is  $\lambda_{4 \rightarrow 2} = 17.2 \text{ nm}$ .
- (d) The longest wavelength that can be emitted is  $\lambda_{2 \rightarrow 1} = 68.7 \text{ nm}$ .
- (e) The second longest wavelength that can be emitted is  $\lambda_{3 \rightarrow 2} = 41.2 \text{ nm}$ .
- (f) The possible transitions are shown next. The energy levels are not drawn to scale.



(g) A wavelength of 29.4 nm corresponds to  $4 \rightarrow 3$  transition. Thus, it could make either the  $3 \rightarrow 1$  transition or the pair of transitions:  $3 \rightarrow 2$  and  $2 \rightarrow 1$ . The longest wavelength that can be emitted is  $\lambda_{2 \rightarrow 1} = 68.7 \text{ nm}$ .

(h) The shortest wavelength that can next be emitted is  $\lambda_{3 \rightarrow 1} = 25.8 \text{ nm}$ .

10. Let the quantum numbers of the pair in question be  $n$  and  $n + 1$ , respectively. Then

$$E_{n+1} - E_n = E_1 (n + 1)^2 - E_1 n^2 = (2n + 1)E_1.$$

Letting

$$E_{n+1} - E_n = (2n + 1)E_1 = 3(E_4 - E_3) = 3(4^2 E_1 - 3^2 E_1) = 21E_1,$$

we get  $2n + 1 = 21$ , or  $n = 10$ . Thus,

(a) the higher quantum number is  $n + 1 = 10 + 1 = 11$ , and

(b) the lower quantum number is  $n = 10$ .

(c) Now letting

$$E_{n+1} - E_n = (2n + 1)E_1 = 2(E_4 - E_3) = 2(4^2 E_1 - 3^2 E_1) = 14E_1,$$

we get  $2n + 1 = 14$ , which does not have an integer-valued solution. So it is impossible to find the pair of energy levels that fits the requirement.

11. Let the quantum numbers of the pair in question be  $n$  and  $n + 1$ , respectively. We note that

$$E_{n+1} - E_n = \frac{(n+1)^2 h^2}{8mL^2} - \frac{n^2 h^2}{8mL^2} = \frac{(2n+1)h^2}{8mL^2}$$

Therefore,  $E_{n+1} - E_n = (2n + 1)E_1$ . Now

$$E_{n+1} - E_n = E_5 = 5^2 E_1 = 25E_1 = (2n+1)E_1,$$

which leads to  $2n + 1 = 25$ , or  $n = 12$ . Thus,

(a) The higher quantum number is  $n + 1 = 12 + 1 = 13$ .

(b) The lower quantum number is  $n = 12$ .

(c) Now let

$$E_{n+1} - E_n = E_6 = 6^2 E_1 = 36E_1 = (2n+1)E_1,$$

which gives  $2n + 1 = 36$ , or  $n = 17.5$ . This is not an integer, so it is impossible to find the pair that fits the requirement.

12. The energy levels are given by  $E_n = n^2 h^2 / 8mL^2$ , where  $h$  is the Planck constant,  $m$  is the mass of an electron, and  $L$  is the width of the well. The frequency of the light that will excite the electron from the state with quantum number  $n_i$  to the state with quantum number  $n_f$  is

$$f = \frac{\Delta E}{h} = \frac{h}{8mL^2} (n_f^2 - n_i^2)$$

and the wavelength of the light is

$$\lambda = \frac{c}{f} = \frac{8mL^2 c}{h(n_f^2 - n_i^2)}.$$

We evaluate this expression for  $n_i = 1$  and  $n_f = 2, 3, 4$ , and  $5$ , in turn. We use  $h = 6.626 \times 10^{-34} \text{ J} \cdot \text{s}$ ,  $m = 9.109 \times 10^{-31} \text{ kg}$ , and  $L = 250 \times 10^{-12} \text{ m}$ , and obtain the following results:

(a)  $6.87 \times 10^{-8} \text{ m}$  for  $n_f = 2$ , (the longest wavelength).

(b)  $2.58 \times 10^{-8} \text{ m}$  for  $n_f = 3$ , (the second longest wavelength).

(c)  $1.37 \times 10^{-8} \text{ m}$  for  $n_f = 4$ , (the third longest wavelength).

13. The position of maximum probability density corresponds to the center of the well:  $x = L/2 = (200 \text{ pm})/2 = 100 \text{ pm}$ .

(a) The probability of detection at  $x$  is given by Eq. 39-11:

$$p(x) = \psi_n^2(x) dx = \left[ \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right) \right]^2 dx = \frac{2}{L} \sin^2\left(\frac{n\pi}{L} x\right) dx$$

For  $n=3$ ,  $L=200 \text{ pm}$ , and  $dx=2.00 \text{ pm}$  (width of the probe), the probability of detection at  $x=L/2=100 \text{ pm}$  is

$$p(x=L/2) = \frac{2}{L} \sin^2\left(\frac{3\pi}{L} \cdot \frac{L}{2}\right) dx = \frac{2}{L} \sin^2\left(\frac{3\pi}{2}\right) dx = \frac{2}{L} dx = \frac{2}{200 \text{ pm}} (2.00 \text{ pm}) = 0.020.$$

(b) With  $N=1000$  independent insertions, the number of times we expect the electron to be detected is  $n = Np = (1000)(0.020) = 20$ .

14. From Eq. 39-11, the condition of zero probability density is given by

$$\sin\left(\frac{n\pi}{L} x\right) = 0 \Rightarrow \frac{n\pi}{L} x = m\pi$$

where  $m$  is an integer. The fact that  $x=0.300L$  and  $x=0.400L$  have zero probability density implies

$$\sin(0.300n\pi) = \sin(0.400n\pi) = 0$$

which can be satisfied for  $n=10m$ , where  $m=1,2,\dots$ . However, since the probability density is nonzero between  $x=0.300L$  and  $x=0.400L$ , we conclude that the electron is in the  $n=10$  state. The change of energy after making a transition to  $n'=9$  is then equal to

$$|\Delta E| = \frac{h^2}{8mL^2} (n^2 - n'^2) = \frac{(6.63 \times 10^{-34} \text{ J}\cdot\text{s})^2}{8(9.11 \times 10^{-31} \text{ kg})(2.00 \times 10^{-10} \text{ m})^2} (10^2 - 9^2) = 2.86 \times 10^{-17} \text{ J}.$$

15. **THINK** The probability that the electron is found in any interval is given by  $P = \int |\psi|^2 dx$ , where the integral is over the interval.

**EXPRESS** If the interval width  $\Delta x$  is small, the probability can be approximated by  $P = |\psi|^2 \Delta x$ , where the wave function is evaluated for the center of the interval, say. For an electron trapped in an infinite well of width  $L$ , the ground state probability density is

$$|\psi|^2 = \frac{2}{L} \sin^2\left(\frac{\pi x}{L}\right),$$

so

$$P = \left(\frac{2\Delta x}{L}\right) \sin^2\left(\frac{\pi x}{L}\right).$$

**ANALYZE** (a) We take  $L = 100$  pm,  $x = 25$  pm, and  $\Delta x = 5.0$  pm. Then,

$$P = \left[\frac{2(5.0 \text{ pm})}{100 \text{ pm}}\right] \sin^2\left[\frac{\pi(25 \text{ pm})}{100 \text{ pm}}\right] = 0.050.$$

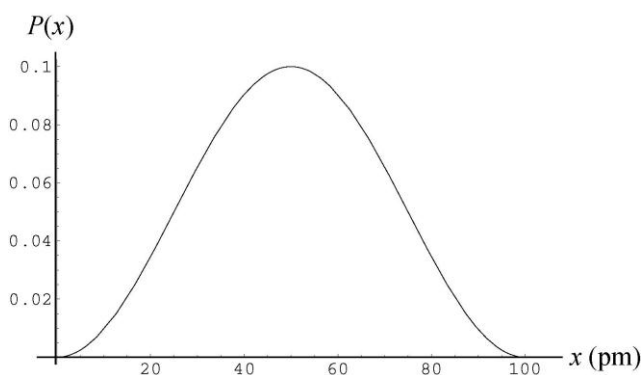
(b) We take  $L = 100$  pm,  $x = 50$  pm, and  $\Delta x = 5.0$  pm. Then,

$$P = \left[\frac{2(5.0 \text{ pm})}{100 \text{ pm}}\right] \sin^2\left[\frac{\pi(50 \text{ pm})}{100 \text{ pm}}\right] = 0.10.$$

(c) We take  $L = 100$  pm,  $x = 90$  pm, and  $\Delta x = 5.0$  pm. Then,

$$P = \left[\frac{2(5.0 \text{ pm})}{100 \text{ pm}}\right] \sin^2\left[\frac{\pi(90 \text{ pm})}{100 \text{ pm}}\right] = 0.0095.$$

**LEARN** The probability as a function of  $x$  is plotted next. As expected, the probability of detecting the electron is highest near the center of the well at  $x = L/2 = 50$  pm.



16. We follow Sample Problem — “Detection potential in a 1D infinite potential well” in the presentation of this solution. The integration result quoted below is discussed in a little more detail in that Sample Problem. We note that the arguments of the sine functions used below are in radians.

(a) The probability of detecting the particle in the region  $0 \leq x \leq L/4$  is

$$\left(\frac{2}{L}\right)\left(\frac{L}{\pi}\right)\int_0^{\pi/4}\sin^2 y \, dy = \frac{2}{\pi}\left(\frac{y}{2} - \frac{\sin 2y}{4}\right)\bigg|_0^{\pi/4} = 0.091.$$

(b) As expected from symmetry,

$$\left(\frac{2}{L}\right)\left(\frac{L}{\pi}\right)\int_{\pi/4}^{\pi}\sin^2 y \, dy = \frac{2}{\pi}\left(\frac{y}{2} - \frac{\sin 2y}{4}\right)\bigg|_{\pi/4}^{\pi} = 0.091.$$

(c) For the region  $L/4 \leq x \leq 3L/4$ , we obtain

$$\left(\frac{2}{L}\right)\left(\frac{L}{\pi}\right)\int_{\pi/4}^{3\pi/4}\sin^2 y \, dy = \frac{2}{\pi}\left(\frac{y}{2} - \frac{\sin 2y}{4}\right)\bigg|_{\pi/4}^{3\pi/4} = 0.82$$

which we could also have gotten by subtracting the results of part (a) and (b) from 1; that is,  $1 - 2(0.091) = 0.82$ .

17. According to Fig. 39-9, the electron's initial energy is 106 eV. After the additional energy is absorbed, the total energy of the electron is 106 eV + 400 eV = 506 eV. Since it is in the region  $x > L$ , its potential energy is 450 eV, so its kinetic energy must be 506 eV - 450 eV = 56 eV.

18. From Fig. 39-9, we see that the sum of the kinetic and potential energies in that particular finite well is 233 eV. The potential energy is zero in the region  $0 < x < L$ . If the kinetic energy of the electron is detected while it is in that region (which is the only region where this is likely to happen), we should find  $K = 233$  eV.

19. Using  $E = hc / \lambda = (1240 \text{ eV} \cdot \text{nm}) / \lambda$ , the energies associated with  $\lambda_a$ ,  $\lambda_b$  and  $\lambda_c$  are

$$E_a = \frac{hc}{\lambda_a} = \frac{1240 \text{ eV} \cdot \text{nm}}{14.588 \text{ nm}} = 85.00 \text{ eV}$$

$$E_b = \frac{hc}{\lambda_b} = \frac{1240 \text{ eV} \cdot \text{nm}}{4.8437 \text{ nm}} = 256.0 \text{ eV}$$

$$E_c = \frac{hc}{\lambda_c} = \frac{1240 \text{ eV} \cdot \text{nm}}{2.9108 \text{ nm}} = 426.0 \text{ eV}.$$

The ground-state energy is

$$E_1 = E_4 - E_c = 450.0 \text{ eV} - 426.0 \text{ eV} = 24.0 \text{ eV}.$$

Since  $E_a = E_2 - E_1$ , the energy of the first excited state is

$$E_2 = E_1 + E_a = 24.0 \text{ eV} + 85.0 \text{ eV} = 109 \text{ eV}.$$



20. The smallest energy a photon can have corresponds to a transition from the non-quantized region to  $E_3$ . Since the energy difference between  $E_3$  and  $E_4$  is

$$\Delta E = E_4 - E_3 = 9.0 \text{ eV} - 4.0 \text{ eV} = 5.0 \text{ eV},$$

the energy of the photon is  $E_{\text{photon}} = K + \Delta E = 2.00 \text{ eV} + 5.00 \text{ eV} = 7.00 \text{ eV}$ .

21. Schrödinger's equation for the region  $x > L$  is

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2m}{h^2} (E - U_0) \psi = 0.$$

If  $\psi = De^{2kx}$ , then  $d^2\psi/dx^2 = 4k^2De^{2kx} = 4k^2\psi$  and

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2m}{h^2} (E - U_0) \psi = 4k^2\psi + \frac{8\pi^2m}{h^2} (E - U_0) \psi.$$

This is zero provided

$$k = \frac{\pi}{h} \sqrt{2m(U_0 - E)}.$$

The proposed function satisfies Schrödinger's equation provided  $k$  has this value. Since  $U_0$  is greater than  $E$  in the region  $x > L$ , the quantity under the radical is positive. This means  $k$  is real. If  $k$  is positive, however, the proposed function is physically unrealistic. It increases exponentially with  $x$  and becomes large without bound. The integral of the probability density over the entire  $x$ -axis must be unity. This is impossible if  $\psi$  is the proposed function.

22. We can use the  $mc^2$  value for an electron from Table 37-3 ( $511 \times 10^3 \text{ eV}$ ) and  $hc = 1240 \text{ eV} \cdot \text{nm}$  by writing Eq. 39-20 as

$$E_{n_x, n_y} = \frac{2h^2}{8m} \left( \frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} \right) = \frac{(hc)^2}{8(mc^2)} \left( \frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} \right).$$

For  $n_x = n_y = 1$ , we obtain

$$E_{1,1} = \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8(511 \times 10^3 \text{ eV})} \left( \frac{1}{(0.800 \text{ nm})^2} + \frac{1}{(1.600 \text{ nm})^2} \right) = 0.734 \text{ eV}.$$

23. We can use the  $mc^2$  value for an electron from Table 37-3 ( $511 \times 10^3 \text{ eV}$ ) and  $hc = 1240 \text{ eV} \cdot \text{nm}$  by writing Eq. 39-21 as

$$E_{n_x, n_y, n_z} = \frac{2h^2}{8m} \left( \frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2} \right) = \frac{(hc)^2}{8(mc^2)} \left( \frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2} \right).$$

For  $n_x = n_y = n_z = 1$ , we obtain

$$E_{1,1} = \frac{(1240 \text{ eV} \cdot \text{nm})^2}{8(511 \times 10^3 \text{ eV})} \left( \frac{1}{(0.800 \text{ nm})^2} + \frac{1}{(1.600 \text{ nm})^2} + \frac{1}{(0.390 \text{ nm})^2} \right) = 3.21 \text{ eV}.$$

24. The statement that there are three probability density maxima along  $x = L_x/2$  implies that  $n_y = 3$  (see for example, Figure 39-6). Since the maxima are separated by 2.00 nm, the width of  $L_y$  is  $L_y = n_y(2.00 \text{ nm}) = 6.00 \text{ nm}$ . Similarly, from the information given along  $y = L_y/2$ , we find  $n_x = 5$  and  $L_x = n_x(3.00 \text{ nm}) = 15.0 \text{ nm}$ . Thus, using Eq. 39-20, the energy of the electron is

$$\begin{aligned} E_{n_x, n_y} &= \frac{h^2}{8m} \left( \frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} \right) = \frac{(6.63 \times 10^{-34} \text{ J} \cdot \text{s})^2}{8(9.11 \times 10^{-31} \text{ kg})} \left[ \frac{1}{(3.00 \times 10^{-9} \text{ m})^2} + \frac{1}{(2.00 \times 10^{-9} \text{ m})^2} \right] \\ &= 2.2 \times 10^{-20} \text{ J}. \end{aligned}$$

25. The discussion on the probability of detection for the one-dimensional case can be readily extended to two dimensions. In analogy to Eq. 39-10, the normalized wave function in two dimensions can be written as

$$\begin{aligned} \psi_{n_x, n_y}(x, y) &= \psi_{n_x}(x) \psi_{n_y}(y) = \sqrt{\frac{2}{L_x}} \sin\left(\frac{n_x \pi}{L_x} x\right) \cdot \sqrt{\frac{2}{L_y}} \sin\left(\frac{n_y \pi}{L_y} y\right) \\ &= \sqrt{\frac{4}{L_x L_y}} \sin\left(\frac{n_x \pi}{L_x} x\right) \sin\left(\frac{n_y \pi}{L_y} y\right). \end{aligned}$$

The probability of detection by a probe of dimension  $\Delta x \Delta y$  placed at  $(x, y)$  is

$$p(x, y) = \left| \psi_{n_x, n_y}(x, y) \right|^2 \Delta x \Delta y = \frac{4(\Delta x \Delta y)}{L_x L_y} \sin^2\left(\frac{n_x \pi}{L_x} x\right) \sin^2\left(\frac{n_y \pi}{L_y} y\right).$$

With  $L_x = L_y = L = 150 \text{ pm}$  and  $\Delta x = \Delta y = 5.00 \text{ pm}$ , the probability of detecting an electron in  $(n_x, n_y) = (1, 3)$  state by placing a probe at  $(0.200L, 0.800L)$  is

$$\begin{aligned}
 p &= \frac{4(\Delta x \Delta y)}{L_x L_y} \sin^2 \left( \frac{n_x \pi}{L_x} x \right) \sin^2 \left( \frac{n_y \pi}{L_y} y \right) = \frac{4(5.00 \text{ pm})^2}{(150 \text{ pm})^2} \sin^2 \left( \frac{\pi}{L} \cdot 0.200L \right) \sin^2 \left( \frac{3\pi}{L} \cdot 0.800L \right) \\
 &= 4 \left( \frac{5.00 \text{ pm}}{150 \text{ pm}} \right)^2 \sin^2 (0.200\pi) \sin^2 (2.40\pi) = 1.4 \times 10^{-3}.
 \end{aligned}$$

26. We are looking for the values of the ratio

$$\frac{E_{n_x, n_y}}{h^2/8mL^2} = L^2 \left( \frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} \right) = \left( n_x^2 + \frac{1}{4} n_y^2 \right)$$

and the corresponding differences.

(a) For  $n_x = n_y = 1$ , the ratio becomes  $1 + \frac{1}{4} = 1.25$ .

(b) For  $n_x = 1$  and  $n_y = 2$ , the ratio becomes  $1 + \frac{1}{4}(4) = 2.00$ . One can check (by computing other  $(n_x, n_y)$  values) that this is the next to lowest energy in the system.

(c) The lowest set of states that are degenerate are  $(n_x, n_y) = (1, 4)$  and  $(2, 2)$ . Both of these states have that ratio equal to  $1 + \frac{1}{4}(16) = 5.00$ .

(d) For  $n_x = 1$  and  $n_y = 3$ , the ratio becomes  $1 + \frac{1}{4}(9) = 3.25$ . One can check (by computing other  $(n_x, n_y)$  values) that this is the lowest energy greater than that computed in part (b). The next higher energy comes from  $(n_x, n_y) = (2, 1)$  for which the ratio is  $4 + \frac{1}{4}(1) = 4.25$ . The difference between these two values is  $4.25 - 3.25 = 1.00$ .

27. **THINK** The energy levels of an electron trapped in a regular corral with widths  $L_x$  and  $L_y$  are given by Eq. 39-20:

$$E_{n_x, n_y} = \frac{h^2}{8m} \left[ \frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} \right].$$

**EXPRESS** With  $L_x = L$  and  $L_y = 2L$ , we have

$$E_{n_x, n_y} = \frac{h^2}{8m} \left[ \frac{n_x^2}{L^2} + \frac{n_y^2}{L^2} \right] = \frac{h^2}{8mL^2} \left[ n_x^2 + \frac{n_y^2}{4} \right].$$

Thus, in units of  $h^2/8mL^2$ , the energy levels are given by  $n_x^2 + n_y^2/4$ . The lowest five levels are  $E_{1,1} = 1.25$ ,  $E_{1,2} = 2.00$ ,  $E_{1,3} = 3.25$ ,  $E_{2,1} = 4.25$ , and  $E_{2,2} = E_{1,4} = 5.00$ . It is clear that there are no other possible values for the energy less than 5.

The frequency of the light emitted or absorbed when the electron goes from an initial state  $i$  to a final state  $f$  is  $f = (E_f - E_i)/h$ , and in units of  $h/8mL^2$  is simply the difference in the values of  $n_x^2 + n_y^2/4$  for the two states. The possible frequencies are as follows:

$$\begin{aligned} &0.75(1, 2 \rightarrow 1, 1), 2.00(1, 3 \rightarrow 1, 1), 3.00(2, 1 \rightarrow 1, 1), \\ &3.75(2, 2 \rightarrow 1, 1), 1.25(1, 3 \rightarrow 1, 2), 2.25(2, 1 \rightarrow 1, 2), 3.00(2, 2 \rightarrow 1, 2), 1.00(2, 1 \rightarrow 1, 3), \\ &1.75(2, 2 \rightarrow 1, 3), 0.75(2, 2 \rightarrow 2, 1), \end{aligned}$$

all in units of  $h/8mL^2$ .

**ANALYZE** (a) From the above, we see that there are 8 different frequencies.

(b) The lowest frequency is, in units of  $h/8mL^2$ , 0.75 (2, 2  $\rightarrow$  2, 1).

(c) The second lowest frequency is, in units of  $h/8mL^2$ , 1.00 (2, 1  $\rightarrow$  1, 3).

(d) The third lowest frequency is, in units of  $h/8mL^2$ , 1.25 (1, 3  $\rightarrow$  1, 2).

(e) The highest frequency is, in units of  $h/8mL^2$ , 3.75 (2, 2  $\rightarrow$  1, 1).

(f) The second highest frequency is, in units of  $h/8mL^2$ , 3.00 (2, 2  $\rightarrow$  1, 2) or (2, 1  $\rightarrow$  1, 1).

(g) The third highest frequency is, in units of  $h/8mL^2$ , 2.25 (2, 1  $\rightarrow$  1, 2).

**LEARN** In general, when the electron makes a transition from  $(n_x, n_y)$  to a higher level  $(n'_x, n'_y)$ , the frequency of photon it emits or absorbs is given by

$$\begin{aligned} f &= \frac{\Delta E}{h} = \frac{E_{n'_x, n'_y} - E_{n_x, n_y}}{h} = \frac{h}{8mL^2} \left( n'^2_x + \frac{n'^2_y}{4} \right) - \frac{h}{8mL^2} \left( n^2_x + \frac{n^2_y}{4} \right) \\ &= \frac{h}{8mL^2} \left[ (n'^2_x - n^2_x) + \frac{1}{4}(n'^2_y - n^2_y) \right]. \end{aligned}$$

28. We are looking for the values of the ratio

$$\frac{E_{n_x, n_y, n_z}}{h^2/8mL^2} = L^2 \left( \frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2} \right) = (n_x^2 + n_y^2 + n_z^2)$$

and the corresponding differences.

(a) For  $n_x = n_y = n_z = 1$ , the ratio becomes  $1 + 1 + 1 = 3.00$ .

(b) For  $n_x = n_y = 2$  and  $n_z = 1$ , the ratio becomes  $4 + 4 + 1 = 9.00$ . One can check (by computing other  $(n_x, n_y, n_z)$  values) that this is the third lowest energy in the system. One can also check that this same ratio is obtained for  $(n_x, n_y, n_z) = (2, 1, 2)$  and  $(1, 2, 2)$ .

(c) For  $n_x = n_y = 1$  and  $n_z = 3$ , the ratio becomes  $1 + 1 + 9 = 11.00$ . One can check (by computing other  $(n_x, n_y, n_z)$  values) that this is three “steps” up from the lowest energy in the system. One can also check that this same ratio is obtained for  $(n_x, n_y, n_z) = (1, 3, 1)$  and  $(3, 1, 1)$ . If we take the difference between this and the result of part (b), we obtain  $11.0 - 9.00 = 2.00$ .

(d) For  $n_x = n_y = 1$  and  $n_z = 2$ , the ratio becomes  $1 + 1 + 4 = 6.00$ . One can check (by computing other  $(n_x, n_y, n_z)$  values) that this is the next to the lowest energy in the system. One can also check that this same ratio is obtained for  $(n_x, n_y, n_z) = (2, 1, 1)$  and  $(1, 2, 1)$ . Thus, three states (three arrangements of  $(n_x, n_y, n_z)$  values) have this energy.

(e) For  $n_x = 1$ ,  $n_y = 2$  and  $n_z = 3$ , the ratio becomes  $1 + 4 + 9 = 14.0$ . One can check (by computing other  $(n_x, n_y, n_z)$  values) that this is five “steps” up from the lowest energy in the system. One can also check that this same ratio is obtained for  $(n_x, n_y, n_z) = (1, 3, 2)$ ,  $(2, 3, 1)$ ,  $(2, 1, 3)$ ,  $(3, 1, 2)$  and  $(3, 2, 1)$ . Thus, six states (six arrangements of  $(n_x, n_y, n_z)$  values) have this energy.

29. The ratios computed in Problem 39-28 can be related to the frequencies emitted using  $f = \Delta E/h$ , where each level  $E$  is equal to one of those ratios multiplied by  $h^2/8mL^2$ . This effectively involves no more than a cancellation of one of the factors of  $h$ . Thus, for a transition from the second excited state (see part (b) of Problem 39-28) to the ground state (treated in part (a) of that problem), we find

$$f = (9.00 - 3.00) \left( \frac{h}{8mL^2} \right) = (6.00) \left( \frac{h}{8mL^2} \right).$$

In the following, we omit the  $h/8mL^2$  factors. For a transition between the fourth excited state and the ground state, we have  $f = 12.00 - 3.00 = 9.00$ . For a transition between the third excited state and the ground state, we have  $f = 11.00 - 3.00 = 8.00$ . For a transition between the third excited state and the first excited state, we have  $f = 11.00 - 6.00 = 5.00$ . For a transition between the fourth excited state and the third excited state, we have  $f = 12.00 - 11.00 = 1.00$ . For a transition between the third excited state and the second excited state, we have  $f = 11.00 - 9.00 = 2.00$ . For a transition between the second excited state and the first excited state, we have  $f = 9.00 - 6.00 = 3.00$ , which also results from some other transitions.

(a) From the above, we see that there are 7 frequencies.

(b) The lowest frequency is, in units of  $h/8mL^2$ , 1.00.

(c) The second lowest frequency is, in units of  $h/8mL^2$ , 2.00.

(d) The third lowest frequency is, in units of  $h/8mL^2$ , 3.00.

(e) The highest frequency is, in units of  $h/8mL^2$ , 9.00.

(f) The second highest frequency is, in units of  $h/8mL^2$ , 8.00.

(g) The third highest frequency is, in units of  $h/8mL^2$ , 6.00.

30. In analogy to Eq. 39-10, the normalized wave function in two dimensions can be written as

$$\begin{aligned}\psi_{n_x, n_y}(x, y) &= \psi_{n_x}(x)\psi_{n_y}(y) = \sqrt{\frac{2}{L_x}} \sin\left(\frac{n_x\pi}{L_x}x\right) \cdot \sqrt{\frac{2}{L_y}} \sin\left(\frac{n_y\pi}{L_y}y\right) \\ &= \sqrt{\frac{4}{L_x L_y}} \sin\left(\frac{n_x\pi}{L_x}x\right) \sin\left(\frac{n_y\pi}{L_y}y\right).\end{aligned}$$

The probability of detection by a probe of dimension  $\Delta x \Delta y$  placed at  $(x, y)$  is

$$p(x, y) = \left| \psi_{n_x, n_y}(x, y) \right|^2 \Delta x \Delta y = \frac{4(\Delta x \Delta y)}{L_x L_y} \sin^2\left(\frac{n_x\pi}{L_x}x\right) \sin^2\left(\frac{n_y\pi}{L_y}y\right).$$

A detection probability of 0.0450 of a ground-state electron ( $n_x = n_y = 1$ ) by a probe of area  $\Delta x \Delta y = 400 \text{ pm}^2$  placed at  $(x, y) = (L/8, L/8)$  implies

$$0.0450 = \frac{4(400 \text{ pm}^2)}{L^2} \sin^2\left(\frac{\pi}{L} \cdot \frac{L}{8}\right) \sin^2\left(\frac{\pi}{L} \cdot \frac{L}{8}\right) = 4\left(\frac{20 \text{ pm}}{L}\right)^2 \sin^4\left(\frac{\pi}{8}\right).$$

Solving for  $L$ , we get  $L = 27.6 \text{ pm}$ .

31. **THINK** The Lyman series is associated with transitions to or from the  $n = 1$  level of the hydrogen atom, while the Balmer series is for transitions to or from the  $n = 2$  level.

**EXPRESS** The energy  $E$  of the photon emitted when a hydrogen atom jumps from a state with principal quantum number  $n'$  to a state with principal quantum number  $n < n'$  is given by

$$E = A \left( \frac{1}{n^2} - \frac{1}{n'^2} \right)$$

where  $A = 13.6 \text{ eV}$ . The frequency  $f$  of the electromagnetic wave is given by  $f = E/h$  and the wavelength is given by  $\lambda = c/f$ . Thus,

$$\frac{1}{\lambda} = \frac{f}{c} = \frac{E}{hc} = \frac{A}{hc} \left( \frac{1}{n^2} - \frac{1}{n'^2} \right).$$

**ANALYZE** The shortest wavelength occurs at the series limit, for which  $n' = \infty$ . For the Balmer series,  $n = 2$  and the shortest wavelength is  $\lambda_B = 4hc/A$ . For the Lyman series,  $n = 1$  and the shortest wavelength is  $\lambda_L = hc/A$ . The ratio is  $\lambda_B/\lambda_L = 4.0$ .

**LEARN** The energy of the photon emitted associated with the transition of an electron from  $n' = \infty \rightarrow n = 2$  (to become bound) is

$$E_{\infty \rightarrow 2} = \frac{13.6 \text{ eV}}{2^2} = 3.4 \text{ eV}.$$

Similarly, the energy associated with the transition of an electron from  $n' = \infty \rightarrow n = 1$  (to become bound) is

$$E_{1 \rightarrow \infty} = \frac{13.6 \text{ eV}}{1^2} = 13.6 \text{ eV}.$$

32. The difference between the energy absorbed and the energy emitted is

$$E_{\text{photon absorbed}} - E_{\text{photon emitted}} = \frac{hc}{\lambda_{\text{absorbed}}} - \frac{hc}{\lambda_{\text{emitted}}}.$$

Thus, using  $hc = 1240 \text{ eV} \cdot \text{nm}$ , the net energy absorbed is

$$hc\Delta\left(\frac{1}{\lambda}\right) = (1240 \text{ eV} \cdot \text{nm}) \left( \frac{1}{375 \text{ nm}} - \frac{1}{580 \text{ nm}} \right) = 1.17 \text{ eV}.$$

33. (a) Since energy is conserved, the energy  $E$  of the photon is given by  $E = E_i - E_f$ , where  $E_i$  is the initial energy of the hydrogen atom and  $E_f$  is the final energy. The electron energy is given by  $(-13.6 \text{ eV})/n^2$ , where  $n$  is the principal quantum number. Thus,

$$E = E_3 - E_1 = \frac{-13.6 \text{ eV}}{(3)^2} - \frac{-13.6 \text{ eV}}{(1)^2} = 12.1 \text{ eV}.$$

(b) The photon momentum is given by

$$p = \frac{E}{c} = \frac{(12.1 \text{ eV})(1.60 \times 10^{-19} \text{ J/eV})}{3.00 \times 10^8 \text{ m/s}} = 6.45 \times 10^{-27} \text{ kg} \cdot \text{m/s}.$$

(c) Using  $hc = 1240 \text{ eV} \cdot \text{nm}$ , the wavelength is  $\lambda = \frac{hc}{E} = \frac{1240 \text{ eV} \cdot \text{nm}}{12.1 \text{ eV}} = 102 \text{ nm}$ .

34. (a) We use Eq. 39-44. At  $r = 0$ ,  $P(r) \propto r^2 = 0$ .

$$(b) \text{ At } r = a, P(r) = \frac{4}{a^3} a^2 e^{-2a/a} = \frac{4e^{-2}}{a} = \frac{4e^{-2}}{5.29 \times 10^{-2} \text{ nm}} = 10.2 \text{ nm}^{-1}.$$

$$(c) \text{ At } r = 2a, P(r) = \frac{4}{a^3} (2a)^2 e^{-4a/a} = \frac{16e^{-4}}{a} = \frac{16e^{-4}}{5.29 \times 10^{-2} \text{ nm}} = 5.54 \text{ nm}^{-1}.$$

35. (a) We use Eq. 39-39. At  $r = a$ ,

$$\psi^2(r) = \left( \frac{1}{\sqrt{\pi} a^{3/2}} e^{-a/a} \right)^2 = \frac{1}{\pi a^3} e^{-2} = \frac{1}{\pi (5.29 \times 10^{-2} \text{ nm})^3} e^{-2} = 291 \text{ nm}^{-3}.$$

(b) We use Eq. 39-44. At  $r = a$ ,

$$P(r) = \frac{4}{a^3} a^2 e^{-2a/a} = \frac{4e^{-2}}{a} = \frac{4e^{-2}}{5.29 \times 10^{-2} \text{ nm}} = 10.2 \text{ nm}^{-1}.$$

36. (a) The energy level corresponding to the probability density distribution shown in Fig. 39-21 is the  $n = 2$  level. Its energy is given by

$$E_2 = -\frac{13.6 \text{ eV}}{2^2} = -3.4 \text{ eV}.$$

(b) As the electron is removed from the hydrogen atom the final energy of the proton-electron system is zero. Therefore, one needs to supply at least 3.4 eV of energy to the system in order to bring its energy up from  $E_2 = -3.4 \text{ eV}$  to zero. (If more energy is supplied, then the electron will retain some kinetic energy after it is removed from the atom.)

37. **THINK** The energy of the hydrogen atom is quantized.

**EXPRESS** If kinetic energy is not conserved, some of the neutron's initial kinetic energy could be used to excite the hydrogen atom. The least energy that the hydrogen atom can accept is the difference between the first excited state ( $n = 2$ ) and the ground state ( $n = 1$ ). Since the energy of a state with principal quantum number  $n$  is  $-(13.6 \text{ eV})/n^2$ , the smallest excitation energy is

$$\Delta E = E_2 - E_1 = \frac{-13.6 \text{ eV}}{(2)^2} - \frac{-13.6 \text{ eV}}{(1)^2} = 10.2 \text{ eV}.$$

**ANALYZE** The neutron, with a kinetic energy of 6.0 eV, does not have sufficient kinetic energy to excite the hydrogen atom, so the hydrogen atom is left in its ground state and



all the initial kinetic energy of the neutron ends up as the final kinetic energies of the neutron and atom. The collision must be elastic.

**LEARN** The minimum kinetic energy the neutron must have in order to excite the hydrogen atom is 10.2 eV.

38. From Eq. 39-6,  $\Delta E = hf = (4.14 \times 10^{-15} \text{ eV} \cdot \text{s})(6.2 \times 10^{14} \text{ Hz}) = 2.6 \text{ eV}$ .

39. **THINK** The radial probability function for the ground state of hydrogen is

$$P(r) = (4r^2/a^3)e^{-2r/a},$$

where  $a$  is the Bohr radius.

**EXPRESS** We want to evaluate the integral  $\int_0^\infty P(r) dr$ . Equation 15 in the integral table of Appendix E is an integral of this form:

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}.$$

**ANALYZE** We set  $n = 2$  and replace  $a$  in the given formula with  $2/a$  and  $x$  with  $r$ . Then

$$\int_0^\infty P(r) dr = \frac{4}{a^3} \int_0^\infty r^2 e^{-2r/a} dr = \frac{4}{a^3} \frac{2}{(2/a)^3} = 1.$$

**LEARN** The integral over the radial probability function  $P(r)$  must be equal to 1. This means that in a hydrogen atom, the electron must be somewhere in the space surrounding the nucleus.

40. (a) The calculation is shown in Sample Problem — “Light emission from a hydrogen atom.” The difference in the values obtained in parts (a) and (b) of that Sample Problem is  $122 \text{ nm} - 91.4 \text{ nm} \approx 31 \text{ nm}$ .

(b) We use Eq. 39-1. For the Lyman series,

$$\Delta f = \frac{2.998 \times 10^8 \text{ m/s}}{91.4 \times 10^{-9} \text{ m}} - \frac{2.998 \times 10^8 \text{ m/s}}{122 \times 10^{-9} \text{ m}} = 8.2 \times 10^{14} \text{ Hz}.$$

(c) Figure 39-18 shows that the width of the Balmer series is  $656.3 \text{ nm} - 364.6 \text{ nm} \approx 292 \text{ nm} \approx 0.29 \mu\text{m}$ .

(d) The series limit can be obtained from the  $\infty \rightarrow 2$  transition:

$$\Delta f = \frac{2.998 \times 10^8 \text{ m/s}}{364.6 \times 10^{-9} \text{ m}} - \frac{2.998 \times 10^8 \text{ m/s}}{656.3 \times 10^{-9} \text{ m}} = 3.65 \times 10^{14} \text{ Hz} \approx 3.7 \times 10^{14} \text{ Hz}.$$

41. Since  $\Delta r$  is small, we may calculate the probability using  $p = P(r) \Delta r$ , where  $P(r)$  is the radial probability density. The radial probability density for the ground state of hydrogen is given by Eq. 39-44:

$$P(r) = \left( \frac{4r^2}{a^3} \right) e^{-2r/a}$$

where  $a$  is the Bohr radius.

(a) Here,  $r = 0.500a$  and  $\Delta r = 0.010a$ . Then,

$$P = \left( \frac{4r^2 \Delta r}{a^3} \right) e^{-2r/a} = 4(0.500)^2 (0.010) e^{-1} = 3.68 \times 10^{-3} \approx 3.7 \times 10^{-3}.$$

(b) We set  $r = 1.00a$  and  $\Delta r = 0.010a$ . Then,

$$P = \left( \frac{4r^2 \Delta r}{a^3} \right) e^{-2r/a} = 4(1.00)^2 (0.010) e^{-2} = 5.41 \times 10^{-3} \approx 5.4 \times 10^{-3}.$$

42. Conservation of linear momentum of the atom-photon system requires that

$$p_{\text{recoil}} = p_{\text{photon}} \Rightarrow m_p v_{\text{recoil}} = \frac{hf}{c}$$

where we use Eq. 39-7 for the photon and use the classical momentum formula for the atom (since we expect its speed to be much less than  $c$ ). Thus, from Eq. 39-6 and Table 37-3,

$$v_{\text{recoil}} = \frac{\Delta E}{m_p c} = \frac{E_4 - E_1}{(m_p c^2)/c} = \frac{(-13.6 \text{ eV})(4^{-2} - 1^{-2})}{(938 \times 10^6 \text{ eV})/(2.998 \times 10^8 \text{ m/s})} = 4.1 \text{ m/s}.$$

43. (a) and (b) Letting  $a = 5.292 \times 10^{-11} \text{ m}$  be the Bohr radius, the potential energy becomes

$$U = -\frac{e^2}{4\pi\epsilon_0 a} = \frac{(8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2)(1.602 \times 10^{-19} \text{ C})^2}{5.292 \times 10^{-11} \text{ m}} = -4.36 \times 10^{-18} \text{ J} = -27.2 \text{ eV}.$$

The kinetic energy is  $K = E - U = (-13.6 \text{ eV}) - (-27.2 \text{ eV}) = 13.6 \text{ eV}$ .

44. (a) Since  $E_2 = -0.85 \text{ eV}$  and  $E_1 = -13.6 \text{ eV} + 10.2 \text{ eV} = -3.4 \text{ eV}$ , the photon energy is

$$E_{\text{photon}} = E_2 - E_1 = -0.85 \text{ eV} - (-3.4 \text{ eV}) = 2.6 \text{ eV}.$$

(b) From

$$E_2 - E_1 = (-13.6 \text{ eV}) \left( \frac{1}{n_2^2} - \frac{1}{n_1^2} \right) = 2.6 \text{ eV}$$

we obtain

$$\frac{1}{n_2^2} - \frac{1}{n_1^2} = \frac{2.6 \text{ eV}}{13.6 \text{ eV}} \approx -\frac{3}{16} = \frac{1}{4^2} - \frac{1}{2^2}.$$

Thus,  $n_2 = 4$  and  $n_1 = 2$ . So the transition is from the  $n = 4$  state to the  $n = 2$  state. One can easily verify this by inspecting the energy level diagram of Fig. 39-18. Thus, the higher quantum number is  $n_2 = 4$ .

(c) The lower quantum number is  $n_1 = 2$ .

45. **THINK** The probability density is given by  $|\psi_{n\ell m_\ell}(r, \theta)|^2$ , where  $\psi_{n\ell m_\ell}(r, \theta)$  is the wave function.

**EXPRESS** To calculate  $|\psi_{n\ell m_\ell}|^2 = \psi_{n\ell m_\ell}^* \psi_{n\ell m_\ell}$ , we multiply the wave function by its complex conjugate. If the function is real, then  $\psi_{n\ell m_\ell}^* = \psi_{n\ell m_\ell}$ . Note that  $e^{+i\phi}$  and  $e^{-i\phi}$  are complex conjugates of each other, and  $e^{i\phi} e^{-i\phi} = e^0 = 1$ .

**ANALYZE** (a)  $\psi_{210}$  is real. Squaring it gives the probability density:

$$|\psi_{210}|^2 = \frac{r^2}{32\pi a^5} e^{-r/a} \cos^2 \theta.$$

(b) Similarly,

$$|\psi_{21+1}|^2 = \frac{r^2}{64\pi a^5} e^{-r/a} \sin^2 \theta$$

and

$$|\psi_{21-1}|^2 = \frac{r^2}{64\pi a^5} e^{-r/a} \sin^2 \theta.$$

The last two functions lead to the same probability density.

(c) For  $m_\ell = 0$ , the probability density  $|\psi_{210}|^2$  decreases with radial distance from the nucleus. With the  $\cos^2 \theta$  factor,  $|\psi_{210}|^2$  is greatest along the  $z$  axis where  $\theta = 0$ . This is consistent with the dot plot of Fig. 39-23(a).

Similarly, for  $m_\ell = \pm 1$ , the probability density  $|\psi_{21\pm 1}|^2$  decreases with radial distance from the nucleus. With the  $\sin^2 \theta$  factor,  $|\psi_{21\pm 1}|^2$  is greatest in the  $xy$ -plane where  $\theta = 90^\circ$ . This is consistent with the dot plot of Fig. 39-23(b).

(d) The total probability density for the three states is the sum:

$$|\psi_{210}|^2 + |\psi_{21+1}|^2 + |\psi_{21-1}|^2 = \frac{r^2}{32\pi a^5} e^{-r/a} \left[ \cos^2 \theta + \frac{1}{2} \sin^2 \theta + \frac{1}{2} \sin^2 \theta \right] = \frac{r^2}{32\pi a^5} e^{-r/a}.$$

The trigonometric identity  $\cos^2 \theta + \sin^2 \theta = 1$  is used. We note that the total probability density does not depend on  $\theta$  or  $\phi$ ; it is spherically symmetric.

**LEARN** The wave functions discussed above are for the hydrogen states with  $n = 2$  and  $\ell = 1$ . Since the angular momentum is nonzero, the probability densities are not spherically symmetric, but depend on both  $r$  and  $\theta$ .

46. From Sample Problem — “Probability of detection of the electron in a hydrogen atom,” we know that the probability of finding the electron in the ground state of the hydrogen atom inside a sphere of radius  $r$  is given by

$$p(r) = 1 - e^{-2x} (1 + 2x + 2x^2)$$

where  $x = r/a$ . Thus the probability of finding the electron between the two shells indicated in this problem is given by

$$\begin{aligned} p(a < r < 2a) &= p(2a) - p(a) = \left[ 1 - e^{-2x} (1 + 2x + 2x^2) \right]_{x=2} - \left[ 1 - e^{-2x} (1 + 2x + 2x^2) \right]_{x=1} \\ &= 0.439. \end{aligned}$$

47. As illustrated in Fig. 39-24, the quantum number  $n$  in question satisfies  $r = n^2 a$ . Letting  $r = 1.0$  nm, we solve for  $n$ :

$$n = \sqrt{\frac{r}{a}} = \sqrt{\frac{1.0 \times 10^{-3} \text{ m}}{5.29 \times 10^{-11} \text{ m}}} \approx 4.3 \times 10^3.$$

48. Using Eq. 39-6 and  $hc = 1240 \text{ eV} \cdot \text{nm}$ , we find

$$\Delta E = E_{\text{photon}} = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{121.6 \text{ nm}} = 10.2 \text{ eV}.$$

Therefore,  $n_{\text{low}} = 1$ , but what precisely is  $n_{\text{high}}$ ?

$$E_{\text{high}} = E_{\text{low}} + \Delta E \Rightarrow -\frac{13.6 \text{ eV}}{n^2} = -\frac{13.6 \text{ eV}}{1^2} + 10.2 \text{ eV}$$

which yields  $n = 2$  (this is confirmed by the calculation found from Sample Problem — “Light emission from a hydrogen atom). Thus, the transition is from the  $n = 2$  to the  $n = 1$  state.

(a) The higher quantum number is  $n = 2$ .

(b) The lower quantum number is  $n = 1$ .

(c) Referring to Fig. 39-18, we see that this must be one of the Lyman series transitions.

49. (a) We take the electrostatic potential energy to be zero when the electron and proton are far removed from each other. Then, the final energy of the atom is zero and the work done in pulling it apart is  $W = -E_i$ , where  $E_i$  is the energy of the initial state. The energy of the initial state is given by  $E_i = (-13.6 \text{ eV})/n^2$ , where  $n$  is the principal quantum number of the state. For the ground state,  $n = 1$  and  $W = 13.6 \text{ eV}$ .

(b) For the state with  $n = 2$ ,  $W = (13.6 \text{ eV})/(2)^2 = 3.40 \text{ eV}$ .

50. Using Eq. 39-6 and  $hc = 1240 \text{ eV} \cdot \text{nm}$ , we find

$$\Delta E = E_{\text{photon}} = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{106.6 \text{ nm}} = 12.09 \text{ eV}.$$

Therefore,  $n_{\text{low}} = 1$ , but what precisely is  $n_{\text{high}}$ ?

$$E_{\text{high}} = E_{\text{low}} + \Delta E \Rightarrow -\frac{13.6 \text{ eV}}{n^2} = -\frac{13.6 \text{ eV}}{1^2} + 12.09 \text{ eV}$$

which yields  $n = 3$ . Thus, the transition is from the  $n = 3$  to the  $n = 1$  state.

(a) The higher quantum number is  $n = 3$ .

(b) The lower quantum number is  $n = 1$ .

(c) Referring to Fig. 39-18, we see that this must be one of the Lyman series transitions.

51. According to Sample Problem — “Probability of detection of the electron in a hydrogen atom,” the probability the electron in the ground state of a hydrogen atom can be found inside a sphere of radius  $r$  is given by

$$p(r) = 1 - e^{-2x}(1 + 2x + 2x^2)$$

where  $x = r/a$  and  $a$  is the Bohr radius. We want  $r = a$ , so  $x = 1$  and

$$p(a) = 1 - e^{-2}(1 + 2 + 2) = 1 - 5e^{-2} = 0.323.$$

The probability that the electron can be found outside this sphere is  $1 - 0.323 = 0.677$ . It can be found outside about 68% of the time.

52. (a)  $\Delta E = -(13.6 \text{ eV})(4^{-2} - 1^{-2}) = 12.8 \text{ eV}.$

(b) There are 6 possible energies associated with the transitions  $4 \rightarrow 3$ ,  $4 \rightarrow 2$ ,  $4 \rightarrow 1$ ,  $3 \rightarrow 2$ ,  $3 \rightarrow 1$  and  $2 \rightarrow 1$ .

(c) The greatest energy is  $E_{4 \rightarrow 1} = 12.8 \text{ eV}.$

(d) The second greatest energy is  $E_{3 \rightarrow 1} = -(13.6 \text{ eV})(3^{-2} - 1^{-2}) = 12.1 \text{ eV}.$

(e) The third greatest energy is  $E_{2 \rightarrow 1} = -(13.6 \text{ eV})(2^{-2} - 1^{-2}) = 10.2 \text{ eV}.$

(f) The smallest energy is  $E_{4 \rightarrow 3} = -(13.6 \text{ eV})(4^{-2} - 3^{-2}) = 0.661 \text{ eV}.$

(g) The second smallest energy is  $E_{3 \rightarrow 2} = -(13.6 \text{ eV})(3^{-2} - 2^{-2}) = 1.89 \text{ eV}.$

(h) The third smallest energy is  $E_{4 \rightarrow 2} = -(13.6 \text{ eV})(4^{-2} - 2^{-2}) = 2.55 \text{ eV}.$

53. **THINK** The ground state of the hydrogen atom corresponds to  $n = 1$ ,  $\ell = 0$ , and  $m_\ell = 0$ .

**EXPRESS** The proposed wave function is

$$\psi = \frac{1}{\sqrt{\pi a^{3/2}}} e^{-r/a}$$

where  $a$  is the Bohr radius. Substituting this into the right side of Schrödinger's equation, our goal is to show that the result is zero.

**ANALYZE** The derivative is

$$\frac{d\psi}{dr} = -\frac{1}{\sqrt{\pi a^{5/2}}} e^{-r/a}$$

so

$$r^2 \frac{d\psi}{dr} = -\frac{r^2}{\sqrt{\pi a^{5/2}}} e^{-r/a}$$

and

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\psi}{dr} \right) = \frac{1}{\sqrt{\pi a^{5/2}}} \left[ -\frac{2}{r} + \frac{1}{a} \right] e^{-r/a} = \frac{1}{a} \left[ -\frac{2}{r} + \frac{1}{a} \right] \psi.$$

The energy of the ground state is given by  $E = -me^4/8\varepsilon_0^2h^2$  and the Bohr radius is given by  $a = h^2\varepsilon_0/\pi me^2$ , so  $E = -e^2/8\pi\varepsilon_0a$ . The potential energy is given by

$$U = -e^2/4\pi\varepsilon_0r,$$

so

$$\begin{aligned} \frac{8\pi^2m}{h^2} E - U \psi &= \frac{8\pi^2m}{h^2} \left[ -\frac{e^2}{8\pi\varepsilon_0a} + \frac{e^2}{4\pi\varepsilon_0r} \right] \psi = \frac{8\pi^2m}{h^2} \frac{e^2}{8\pi\varepsilon_0} \left[ -\frac{1}{a} + \frac{2}{r} \right] \psi \\ &= \frac{\pi me^2}{h^2\varepsilon_0} \left[ -\frac{1}{a} + \frac{2}{r} \right] \psi = \frac{1}{a} \left[ -\frac{1}{a} + \frac{2}{r} \right] \psi. \end{aligned}$$

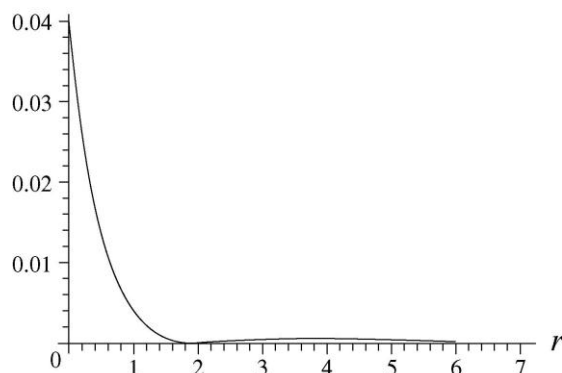
The two terms in Schrödinger's equation cancel, and the proposed function  $\psi$  satisfies that equation.

**LEARN** The radial probability density of the ground state of hydrogen atom is given by Eq. 39-44:

$$P(r) = |\psi|^2 (4\pi r^2) = \frac{1}{\pi a^3} e^{-2r/a} (4\pi r^2) = \frac{4}{a^3} r^2 e^{-2r/a}.$$

A plot of  $P(r)$  is shown in Fig. 39-20.

54. (a) The plot shown below for  $|\psi_{200}(r)|^2$  is to be compared with the dot plot of Fig. 39-21. We note that the horizontal axis of our graph is labeled “ $r$ ,” but it is actually  $r/a$  (that is, it is in units of the parameter  $a$ ). Now, in the plot below there is a high central peak between  $r = 0$  and  $r \sim 2a$ , corresponding to the densely dotted region around the center of the dot plot of Fig. 39-21. Outside this peak is a region of near-zero values centered at  $r = 2a$ , where  $\psi_{200} = 0$ . This is represented in the dot plot by the empty ring surrounding the central peak. Further outside is a broader, flatter, low peak that reaches its maximum value at  $r = 4a$ . This corresponds to the outer ring with near-uniform dot density, which is lower than that of the central peak.



(b) The extrema of  $\psi^2(r)$  for  $0 < r < \infty$  may be found by squaring the given function, differentiating with respect to  $r$ , and setting the result equal to zero:

$$-\frac{1}{32} \frac{(r-2a)(r-4a)}{a^6 \pi} e^{-r/a} = 0$$

which has roots at  $r = 2a$  and  $r = 4a$ . We can verify directly from the plot above that  $r = 4a$  is indeed a local maximum of  $\psi_{200}^2(r)$ . As discussed in part (a), the other root ( $r = 2a$ ) is a local minimum.

(c) Using Eq. 39-43 and Eq. 39-41, the radial probability is

$$P_{200}(r) = 4\pi r^2 \psi_{200}^2(r) = \frac{r^2}{8a^3} \left(2 - \frac{r}{a}\right)^2 e^{-r/a}.$$

(d) Let  $x = r/a$ . Then

$$\begin{aligned} \int_0^\infty P_{200}(r) dr &= \int_0^\infty \frac{r^2}{8a^3} \left(2 - \frac{r}{a}\right)^2 e^{-r/a} dr = \frac{1}{8} \int_0^\infty x^2 (2-x)^2 e^{-x} dx = \int_0^\infty (x^4 - 4x^3 + 4x^2) e^{-x} dx \\ &= \frac{1}{8} [4! - 4(3!) + 4(2!)] = 1 \end{aligned}$$

where we have used the integral formula  $\int_0^\infty x^n e^{-x} dx = n!$ .

55. The radial probability function for the ground state of hydrogen is

$$P(r) = (4r^2/a^3) e^{-2r/a},$$

where  $a$  is the Bohr radius. (See Eq. 39-44.) The integral table of Appendix E may be used to evaluate the integral  $r_{\text{avg}} = \int_0^\infty rP(r) dr$ . Setting  $n = 3$  and replacing  $a$  in the given formula with  $2/a$  (and  $x$  with  $r$ ), we obtain

$$r_{\text{avg}} = \int_0^\infty rP(r) dr = \frac{4}{a^3} \int_0^\infty r^3 e^{-2r/a} dr = \frac{4}{a^3} \frac{6}{(2/a)^4} = 1.5a.$$

56. (a) The allowed energy values are given by  $E_n = n^2 h^2 / 8mL^2$ . The difference in energy between the state  $n$  and the state  $n + 1$  is

$$\Delta E_{\text{adj}} = E_{n+1} - E_n = (n+1)^2 \frac{h^2}{8mL^2} - n^2 \frac{h^2}{8mL^2} = \frac{(2n+1)h^2}{8mL^2}$$

and



$$\frac{\Delta E_{\text{adj}}}{E} = \left[ \frac{(2n+1)h^2}{8mL^2} \right] \left( \frac{8mL^2}{n^2 h^2} \right) = \frac{2n+1}{n^2}.$$

As  $n$  becomes large,  $2n+1 \rightarrow 2n$  and  $(2n+1)/n^2 \rightarrow 2n/n^2 = 2/n$ .

(b) No. As  $n \rightarrow \infty$ ,  $\Delta E_{\text{adj}}$  and  $E$  do not approach 0, but  $\Delta E_{\text{adj}}/E$  does.

(c) No. See part (b).

(d) Yes. See part (b).

(e)  $\Delta E_{\text{adj}}/E$  is a better measure than either  $\Delta E_{\text{adj}}$  or  $E$  alone of the extent to which the quantum result is approximated by the classical result.

57. From Eq. 39-4,

$$E_{n+2} - E_n = \left( \frac{h^2}{8mL^2} \right) (n+2)^2 - \left( \frac{h^2}{8mL^2} \right) n^2 = \left( \frac{h^2}{2mL^2} \right) (n+1).$$

58. (a) and (b) In the region  $0 < x < L$ ,  $U_0 = 0$ , so Schrödinger's equation for the region is

$$\frac{d^2 \psi}{dx^2} + \frac{8\pi^2 m}{h^2} E \psi = 0$$

where  $E > 0$ . If  $\psi^2(x) = B \sin^2 kx$ , then  $\psi(x) = B' \sin kx$ , where  $B'$  is another constant satisfying  $B'^2 = B$ . Thus,

$$\frac{d^2 \psi}{dx^2} = -k^2 B' \sin kx = -k^2 \psi(x)$$

and

$$\frac{d^2 \psi}{dx^2} + \frac{8\pi^2 m}{h^2} E \psi = -k^2 \psi + \frac{8\pi^2 m}{h^2} E \psi.$$

This is zero provided that  $k^2 = \frac{8\pi^2 m E}{h^2}$ . The quantity on the right-hand side is positive, so  $k$  is real and the proposed function satisfies Schrödinger's equation. In this case, there exists no physical restriction as to the sign of  $k$ . It can assume either positive or negative values. Thus,  $k = \pm \frac{2\pi}{h} \sqrt{2mE}$ .

59. **THINK** For a finite well, the electron matter wave can penetrate the walls of the well. Thus, the wave function outside the well is not zero, but decreases exponentially with distance.

**EXPRESS** Schrödinger's equation for the region  $x > L$  is

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2m}{h^2} (E - U_0) \psi = 0,$$

where  $E - U_0 < 0$ . If  $\psi^2(x) = Ce^{-2kx}$ , then  $\psi(x) = \sqrt{C} e^{-kx}$ .

**ANALYZE** (a) and (b) Thus,

$$\frac{d^2\psi}{dx^2} = 4k^2 \sqrt{C} e^{-kx} = 4k^2 \psi$$

and

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2m}{h^2} (E - U_0) \psi = k^2 \psi + \frac{8\pi^2m}{h^2} (E - U_0) \psi.$$

This is zero provided that  $k^2 = \frac{8\pi^2m}{h^2} (U_0 - E)$ . Choosing the positive root, we have

$$k = \frac{2\pi}{h} \sqrt{2m(U_0 - E)}.$$

**LEARN** Note that the quantity  $U_0 - E$  is positive, so  $k$  is real and the proposed function satisfies Schrödinger's equation. If  $k$  is negative, however, the proposed function would be physically unrealistic. It would increase exponentially with  $x$ . Since the integral of the probability density over the entire  $x$  axis must be finite,  $\psi$  diverging as  $x \rightarrow \infty$  would be unacceptable.

60. We can use the  $mc^2$  value for an electron from Table 37-3 ( $511 \times 10^3$  eV) and  $hc = 1240$  eV · nm by writing Eq. 39-4 as

$$E_n = \frac{n^2 h^2}{8mL^2} = \frac{n^2 (hc)^2}{8(mc^2)L^2}.$$

(a) With  $L = 3.0 \times 10^9$  nm, the energy difference is

$$E_2 - E_1 = \frac{1240^2}{8(511 \times 10^3)(3.0 \times 10^9)^2} (2^2 - 1^2) = 1.3 \times 10^{-19} \text{ eV}.$$

(b) Since  $(n+1)^2 - n^2 = 2n+1$ , we have

$$\Delta E = E_{n+1} - E_n = \frac{h^2}{8mL^2} (2n+1) = \frac{(hc)^2}{8(mc^2)L^2} (2n+1).$$

Setting this equal to 1.0 eV, we solve for  $n$ :

$$n = \frac{4(mc^2)L^2\Delta E}{(hc)^2} - \frac{1}{2} = \frac{4(511 \times 10^3 \text{ eV})(3.0 \times 10^9 \text{ nm})^2(1.0 \text{ eV})}{(1240 \text{ eV} \cdot \text{nm})^2} - \frac{1}{2} \approx 1.2 \times 10^{19}.$$

(c) At this value of  $n$ , the energy is

$$E_n = \frac{1240^2}{8(511 \times 10^3)(3.0 \times 10^9)^2} (6 \times 10^{18})^2 \approx 6 \times 10^{18} \text{ eV}.$$

Thus,

$$\frac{E_n}{mc^2} = \frac{6 \times 10^{18} \text{ eV}}{511 \times 10^3 \text{ eV}} = 1.2 \times 10^{13}.$$

(d) Since  $E_n / mc^2 \gg 1$ , the energy is indeed in the relativistic range.

61. (a) We recall that a derivative with respect to a dimensional quantity carries the (reciprocal) units of that quantity. Thus, the first term in Eq. 39-18 has dimensions of  $\psi$  multiplied by dimensions of  $x^{-2}$ . The second term contains no derivatives, does contain  $\psi$ , and involves several other factors that turn out to have dimensions of  $x^{-2}$ :

$$\frac{8\pi^2 m}{h^2} [E - U(x)] \Rightarrow \frac{\text{kg}}{(\text{J} \cdot \text{s})^2} [\text{J}]$$

assuming SI units. Recalling from Eq. 7-9 that  $\text{J} = \text{kg} \cdot \text{m}^2/\text{s}^2$ , then we see the above is indeed in units of  $\text{m}^{-2}$  (which means dimensions of  $x^{-2}$ ).

(b) In one-dimensional quantum physics, the wave function has units of  $\text{m}^{-1/2}$ , as shown in Eq. 39-17. Thus, since each term in Eq. 39-18 has units of  $\psi$  multiplied by units of  $x^{-2}$ , then those units are  $\text{m}^{-1/2} \cdot \text{m}^{-2} = \text{m}^{-2.5}$ .

62. (a) The “home-base” energy level for the Balmer series is  $n = 2$ . Thus the transition with the least energetic photon is the one from the  $n = 3$  level to the  $n = 2$  level. The energy difference for this transition is

$$\Delta E = E_3 - E_2 = -(13.6 \text{ eV}) \left( \frac{1}{3^2} - \frac{1}{2^2} \right) = 1.889 \text{ eV}.$$

Using  $hc = 1240 \text{ eV} \cdot \text{nm}$ , the corresponding wavelength is

$$\lambda = \frac{hc}{\Delta E} = \frac{1240 \text{ eV} \cdot \text{nm}}{1.889 \text{ eV}} = 658 \text{ nm}.$$

(b) For the series limit, the energy difference is

$$\Delta E = E_{\infty} - E_2 = -(13.6\text{eV})\left(\frac{1}{\infty^2} - \frac{1}{2^2}\right) = 3.40\text{eV}.$$

The corresponding wavelength is then  $\lambda = \frac{hc}{\Delta E} = \frac{1240\text{eV}\cdot\text{nm}}{3.40\text{eV}} = 366\text{nm}$ .

63. (a) The allowed values of  $\ell$  for a given  $n$  are 0, 1, 2, ...,  $n - 1$ . Thus there are  $n$  different values of  $\ell$ .

(b) The allowed values of  $m_{\ell}$  for a given  $\ell$  are  $-\ell, -\ell + 1, \dots, \ell$ . Thus there are  $2\ell + 1$  different values of  $m_{\ell}$ .

(c) According to part (a) above, for a given  $n$  there are  $n$  different values of  $\ell$ . Also, each of these  $\ell$ 's can have  $2\ell + 1$  different values of  $m_{\ell}$  [see part (b) above]. Thus, the total number of  $m_{\ell}$ 's is

$$\sum_{\ell=0}^{n-1} (2\ell + 1) = n^2.$$

64. For  $n = 1$

$$E_1 = -\frac{m_e e^4}{8\epsilon_0^2 h^2} = -\frac{(9.11 \times 10^{-31}\text{ kg})(1.6 \times 10^{-19}\text{ C})^4}{8(8.85 \times 10^{-12}\text{ F/m})^2 (6.63 \times 10^{-34}\text{ J}\cdot\text{s})^2 (1.60 \times 10^{-19}\text{ J/eV})} = -13.6\text{eV}.$$

65. (a) The angular momentum of the diatomic gas is

$$L = I\omega = 2 \times m(d/2)^2 \omega = \frac{1}{2}md^2\omega.$$

If its angular momentum is quantized, i.e., restricted to  $L = n\hbar$ ,  $n = 1, 2, \dots$  then

$$\frac{1}{2}md^2\omega = n\hbar = \frac{nh}{2\pi} \Rightarrow \omega = \frac{nh}{\pi md^2}$$

(b) The quantized rotational energies are

$$E_n = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{md^2}{2}\right)\left(\frac{nh}{\pi md^2}\right)^2 = \frac{n^2\hbar^2}{4\pi^2 md^2}$$

66. The expression for the probability of detecting an electron in the ground state of hydrogen atom inside a sphere of radius  $r$  is given in Sample Problem 39.07:

$$p(x) = 1 - e^{-2x}(1 + 2x + 2x^2)$$

where  $x = r/a_0$ , with  $a_0 = 5.292 \times 10^{-11} \text{ m}$ . Given that  $r = 1.1 \times 10^{-15} \text{ m}$ ,

$$x = (1.1 \times 10^{-15} \text{ m}) / (5.292 \times 10^{-11} \text{ m}) = 2.079 \times 10^{-5}.$$

For small  $x$ ,  $p(x)$  can be simplified as

$$\begin{aligned} p(x) &= 1 - e^{-2x}(1 + 2x + 2x^2) \approx 1 - \left(1 - 2x + 2x^2 - \frac{4}{3}x^3 + \dots\right)(1 + 2x + 2x^2) = \frac{4}{3}x^3 \\ &= \frac{4}{3}(2.079 \times 10^{-5})^3 = 1.2 \times 10^{-14}. \end{aligned}$$

67. (a) For a particle of mass  $m$  trapped inside a container of length  $L$ , the allowed energy values are given by  $E_n = n^2 h^2 / 8mL^2$ . With an argon atom and  $L = 0.20 \text{ m}$ , the energy difference between the lowest two levels is

$$\begin{aligned} \Delta E &= E_2 - E_1 = \frac{h^2}{8mL^2}(2^2 - 1^2) = \frac{3h^2}{8mL^2} = \frac{3(6.626 \times 10^{-34} \text{ J}\cdot\text{s})^2}{8(0.0399 \text{ kg}/6.02 \times 10^{23})(0.20 \text{ m})^2} \\ &= 6.21 \times 10^{-41} \text{ J} = 3.88 \times 10^{-22} \text{ eV}. \end{aligned}$$

(b) The thermal energy at  $T = 300 \text{ K}$  is its average kinetic energy:

$$\bar{K} = \frac{3}{2}kT = (1.38 \times 10^{-23} \text{ J/K})(300 \text{ K}) = 6.21 \times 10^{-21} \text{ J} = 3.88 \times 10^{-2} \text{ eV}.$$

Thus, the ratio is

$$\frac{\bar{K}}{\Delta E} = \frac{3.88 \times 10^{-2} \text{ eV}}{3.9 \times 10^{-22} \text{ eV}} = 10^{20}.$$

(c) The temperature at which  $\bar{K} = \frac{3}{2}kT = \Delta E$  is

$$T = \frac{2(\Delta E)}{3k} = \frac{2(6.21 \times 10^{-41} \text{ J})}{3(1.38 \times 10^{-23} \text{ J/K})} = 3.0 \times 10^{-18} \text{ K}.$$

68. The muon orbits the  $\text{He}^+$  nucleus at a speed given by ( $k = 1/4\pi\epsilon_0$ )

$$\frac{mv^2}{r} = \frac{Zke^2}{r^2} \Rightarrow v = \sqrt{\frac{Zke^2}{mr}}$$

With quantization condition  $L = mvr = n\hbar$ , the allowed values of the radius is

$$r_n = \frac{n^2 \hbar^2}{Zke^2 m}$$

Its total energy is

$$E = K + U = \frac{1}{2}mv^2 - \frac{Zke^2}{r} = -\frac{Zke^2}{2r}$$

The energy of the muon ground state is given by

$$E_n = -\frac{Zke^2}{2r_n} = -\frac{m(Ze^2)^2}{8\epsilon_0^2 \hbar^2} \frac{1}{n^2}$$

Evaluating the constants gives

$$\begin{aligned} E_n &= -\frac{m(Ze^2)^2}{8\epsilon_0^2 \hbar^2} \frac{1}{n^2} = -\frac{(207 \times 9.11 \times 10^{-31} \text{ kg})(2)^2 (1.6 \times 10^{-19} \text{ C})^4}{8(8.85 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2)^2 (6.626 \times 10^{-34} \text{ J} \cdot \text{s})^2} \frac{1}{n^2} \\ &= -\frac{1.8 \times 10^{-15} \text{ J}}{n^2} = -\frac{11.3 \text{ keV}}{n^2}. \end{aligned}$$

69. The Ritz combination principle can be readily understood by noting that the transition from  $n = n_i$  to  $n = n_f < n_i$  can be done in two steps, with an intermediate state  $n'$ :

$$\Delta E = E_{n_f} - E_{n_i} = (-13.6 \text{ eV}) \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right) = (-13.6 \text{ eV}) \left( \frac{1}{n_f^2} - \frac{1}{n'^2} \right) + (-13.6 \text{ eV}) \left( \frac{1}{n'^2} - \frac{1}{n_i^2} \right)$$

The transition  $n_i = 3 \rightarrow n_f = 1$  associated with the second Lyman-series line can be thought of as  $n_i = 3 \rightarrow n' = 2$  (first Balmer) followed by  $n' = 2 \rightarrow n_f = 1$  (first Lyman). Another example would be  $n_i = 4 \rightarrow n_f = 2$  (second Balmer), which can be thought of as  $n_i = 4 \rightarrow n' = 3$  (first Paschen) followed by  $n' = 3 \rightarrow n_f = 2$  (first Balmer).

70. (a) We use  $e_0$  to denote the electric charge. The constant  $A$  can be calculated by integrating the charge density distribution:

$$-e_0 = \int \rho(r) dV = \int_0^\infty (Ae^{-2r/a_0}) 4\pi r^2 dr = 4\pi A a_0^3 \int_0^\infty x^2 e^{-2x} dx = \pi A a_0^3$$

which gives  $A = -e_0 / \pi a_0^3$ .

(b) We apply Gauss's to calculate the electric field at a distance  $r$  from the center of the atom. The charge enclosed by a Gaussian sphere of radius  $r = a_0$ , including the proton charge  $+e_0$  at the center, is

$$\begin{aligned}
 q_{\text{enc}} &= e_0 + \int \rho(r) dV = e_0 + \int_0^{a_0} (Ae^{-2r/a_0}) 4\pi r^2 dr = e_0 + 4\pi A a_0^3 \int_0^1 x^2 e^{-2x} dx \\
 &= e_0 + \pi A a_0^3 \left(1 - \frac{5}{e^2}\right) = e_0 + (-e_0) \left(1 - \frac{5}{e^2}\right) = (5e^{-2})e_0
 \end{aligned}$$

Using Gauss's law,  $\int \vec{E} \cdot d\vec{a} = q_{\text{enc}} / \epsilon_0$ , we obtain

$$E(4\pi a_0^2) = \frac{(5e^{-2})e_0}{\epsilon_0} \Rightarrow E = \frac{(5e^{-2})e_0}{4\pi\epsilon_0 a_0^2}$$

(c) The net charge enclosed is positive, so the direction is radially outward.

71. (a) The charge enclosed by a sphere of radius  $r$  due to the uniform positive charge distribution is proportional to the volume:  $q_{\text{enc}} = e(r/a_0)^3$ . Using Gauss's law,

$\int \vec{E} \cdot d\vec{a} = q_{\text{enc}} / \epsilon_0$ , the electric field at a radial distance  $r$  from the center of the atom is

$$E(4\pi r^2) = \frac{e}{\epsilon_0} \left(\frac{r}{a_0}\right)^3 \Rightarrow E = \frac{e}{4\pi\epsilon_0 a_0^3} r$$

and the force on the electron is  $F = -eE = \frac{-e^2}{4\pi\epsilon_0 a_0^3} r$ . The negative sign means that the force points toward the center.

(b) Since  $F = ma = md^2r/dt^2$ ,

$$m \frac{d^2r}{dt^2} = \frac{-e^2}{4\pi\epsilon_0 a_0^3} r \Rightarrow \frac{d^2r}{dt^2} + \omega^2 r = 0$$

and the angular frequency is

$$\omega = \sqrt{\frac{e^2}{4\pi\epsilon_0 m a_0^3}} = \frac{e}{\sqrt{4\pi\epsilon_0 m a_0^3}}.$$

72. (a) The electric potential is

$$V = \frac{kq}{r} = \frac{ke}{a_0} = \frac{8.99 \times 10^9 \text{ N} \cdot \text{m}^2 / \text{C}^2}{5.29 \times 10^{-11} \text{ m}} = 27.22 \text{ V}$$

(b) The electric potential energy of the atom is

$$U = qV = -eV = -e(27.22 \text{ V}) = -27.22 \text{ eV}$$

(c) The electron moves in a circular orbit with

$$\frac{mv^2}{r} = \frac{ke^2}{r^2} \Rightarrow v = \sqrt{\frac{ke^2}{mr}}$$

Its kinetic energy at  $r = a_0$  is

$$K = \frac{1}{2}mv^2 = \frac{ke^2}{2a_0} = \frac{1}{2}(27.22 \text{ eV}) = 13.6 \text{ eV}.$$

(d) The total energy of the system is

$$E = K + U = \frac{1}{2}mv^2 - \frac{ke^2}{a_0} = -\frac{ke^2}{2a_0} = -13.6 \text{ eV}.$$

Therefore, the energy required to ionize the atom is +13.6 eV.

73. The energy is, after evaluating the constants,

$$\begin{aligned} E_{n_1, n_2, n_3} &= \frac{h^2}{8mL^2} (n_1^2 + n_2^2 + n_3^2) = \frac{(6.626 \times 10^{-34} \text{ J} \cdot \text{s})^2}{8(9.11 \times 10^{-31} \text{ kg})(0.25 \times 10^{-6} \text{ m})^2} (n_1^2 + n_2^2 + n_3^2) \\ &= (6.024 \text{ } \mu\text{eV}) (n_1^2 + n_2^2 + n_3^2) \end{aligned}$$

The lowest five states correspond to  $(n_1, n_2, n_3) = (1, 1, 1), (1, 2, 1), (1, 2, 2), (1, 3, 1)$  and  $(2, 2, 2)$ , and the energies are

$$\begin{aligned} E_{111} &= \frac{h^2}{8mL^2} (1^2 + 1^2 + 1^2) = 3(6.024 \text{ } \mu\text{eV}) = 18.1 \text{ } \mu\text{eV} \\ E_{121} &= \frac{h^2}{8mL^2} (1^2 + 2^2 + 1^2) = 6(6.024 \text{ } \mu\text{eV}) = 36.2 \text{ } \mu\text{eV} \\ E_{122} &= \frac{h^2}{8mL^2} (1^2 + 2^2 + 2^2) = 9(6.024 \text{ } \mu\text{eV}) = 54.3 \text{ } \mu\text{eV} \\ E_{131} &= \frac{h^2}{8mL^2} (1^2 + 3^2 + 1^2) = 11(6.024 \text{ } \mu\text{eV}) = 66.3 \text{ } \mu\text{eV} \\ E_{222} &= \frac{h^2}{8mL^2} (2^2 + 2^2 + 2^2) = 12(6.024 \text{ } \mu\text{eV}) = 72.4 \text{ } \mu\text{eV} \end{aligned}$$